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Coupling parameter in synchronization of diluted neural networks

Qi Li,[1] Yong Chen,[1,2,*] and Ying Hai Wang[1]
[1]*Department of Physics, Lanzhou University, Gansu 730000, China*
[2]*State Key Laboratory of Frozen Soil Engineering, CAREERI, CAS, Lanzhou 730000, China*

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We study the critical features of a coupling parameter in the synchronization of neural networks with diluted synapses. Based on simulations, an exponential decay form is observed in the extreme case of global coupling among subsystems and full connection in each network: there exists a maximum and a minimum of the critical coupling intensity for synchronization in this spatially extended system. For partial coupling, we present the primary result about the critical coupling fraction for various linking degrees of the networks.

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Synchronization of coupled complex systems has been an intensively studied subject since the pioneering work of Fujisaka [1] and others [2]. This phenomenon of synchronization is observed in many other fields, such as in neural networks [3], biological populations [4], and chemical reactions [5]. Recently, spatially extended systems have inspired great interest [6].

Following the series of works contributed by Zanette and co-workers [6–8], we consider a simple modified version of the neural network model described in [8]. As a very important ubiquitous factor corresponding to real biotic neural systems, the neural activity and morphology of synaptic connectivity, i.e., the dilution of neural networks must be introduced [9]. That is, in natural neural systems, not all the neurons are linked together. So, there is a chance to investigate the critical features of the coupling parameter and the function of structural topology in the synchronization of extended systems.

We consider a neural network model that consists of N analog neurons $x_i(t) \in (0,1),\ i=1,\dots,N$. Each neuron x_i is connected with other neurons x_j by a random weighted coupling $J_{ij} \in (-1,1)$, which is a randomly independent variable with uniform distribution. Obviously, the connectivity matrix $\mathbf{J}$ is asymmetric and the neural system exhibits complex spatial oscillations. As a simple version of coupled neural systems designed by Zanette and Mikhailov [8], we use the following parallel dynamics for the updating neurons:

$$x_i^1(t+1) = (1-\varepsilon)\Theta(h_i^1(t)) + \varepsilon\Theta(h_i^1(t)+h_i^2(t)),$$

$$x_i^2(t+1) = (1-\varepsilon)\Theta(h_i^2(t)) + \varepsilon\Theta(h_i^1(t)+h_i^2(t)). \quad (1)$$

Here $h_i^k(t)$ is the local field of the ith neuron and is expressed by

$$h_i^k(t) = \sum_{j=1}^{N} C_{ij} J_{ij} x_j^k(t), \quad (2)$$

where $C_{ij} \in \{0,1\}$ is used to denote the linked status between the ith and the jth neuron. The dilution factor C_{ij} is an independent identically distributed random variable, which is selected by $C_{ij}=1$ with the connection probability $d \in [0,1]$, or $C_{ij}=0$ with probability $1-d$ [10]. The activation function $\Theta(r)$ is defined as $\Theta(r)=[1+\tanh(\beta r)]/2$. Here, $\beta \equiv 1/T$ characterizes a measure of the inverse magnitude of the amount of noise affecting this neuron, performing the role of reciprocal temperature in analogy to thermodynamic systems. For convenience, we set $\beta=10$ through all simulations.

Obviously, the first term on the right-hand side of Eq. (1) pictures the total responses from its own internal units. The second term expresses the interaction of the summation of the received signals from the neurons at the same position in two networks. The parameter $\varepsilon \in [0,1]$, called the coupling intensity, describes the interaction degree between coupling subsystems. When the intensity $\varepsilon \ll 1$, it is easy to see that the coupling subsystems evolve independently. On the other limit case, $\varepsilon \approx 1$, the coupling subsystems are governed by the same dynamical law and will be driven into synchronization very easily.

For measuring coherence in the collective activity of the neural systems, an important time-dependent feature, an activity function $u_i(t)=\Sigma_{k=1}^{N} x_k^i(t)$ for each network $i=1,2$, is introduced. When the global coupling of two systems is absent, $\varepsilon \approx 0$, $u_1(t)$ and $u_2(t)$ will update independently and not become synchronized since the initial conditions in the subsystems are different. On the other hand, the activity signals of two subsystems will be identical if the coupling systems become synchronous. Figure 1(a) shows that the synchronization for this extended system successfully takes place at $t=275$ for both subsystems with $N=100$, while the coupling intensity $\varepsilon=0.34$ and the linked degree in each subsystem $d=0.2$.

To show the degree of synchronization in this coupling system, the dispersion of activity patterns is defined as

$$D(t) = \frac{1}{2}\sum_{i=1}^{2}\sum_{k=1}^{N}[x_k^i(t)-\overline{x_k}(t)]^2, \quad (3)$$

where $\overline{x_k}(t)=2^{-1}\Sigma_{i=1}^{2} x_k^i(t)$ denotes the average activity of neurons occupying the kth position in both subsystems at time t. Figure 1(b) shows that the dispersion with a logarithmic scale evolves in time, with the same synchronous conditions in Fig. 1(a).

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[*] Corresponding author. Email address: ychen@lzu.edu.cn

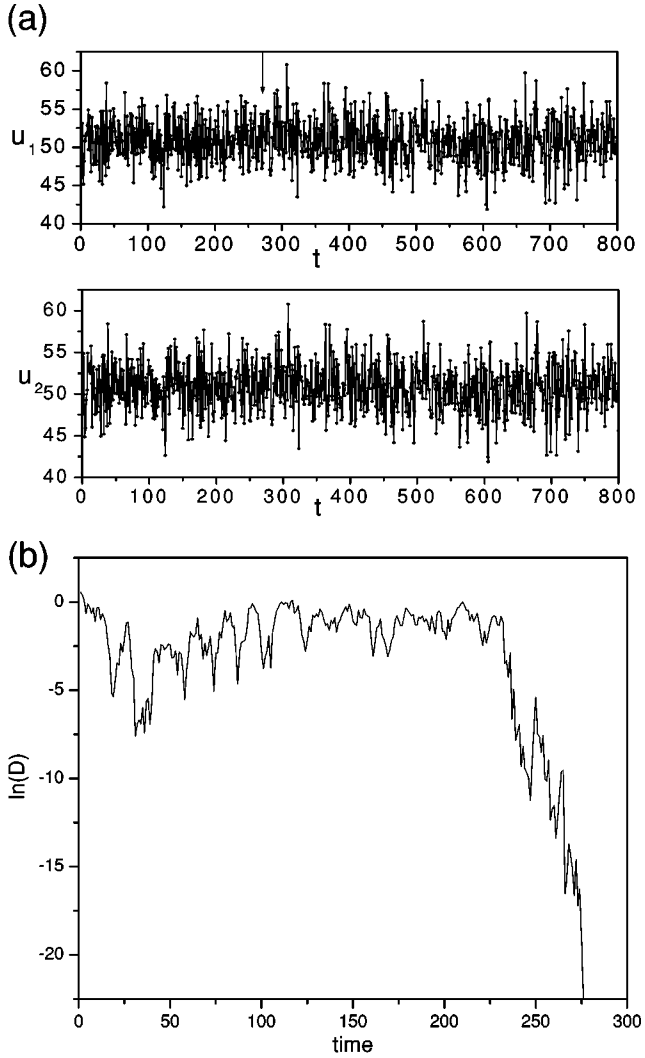

FIG. 1. The synchronization of diluted networks with $N=100$, $\varepsilon=0.34$, and $d=0.2$. (a) The evolution of time-dependent activity of both subsystems comes into synchronization at $t=275$. (b) Logarithm of dispersion of both networks.

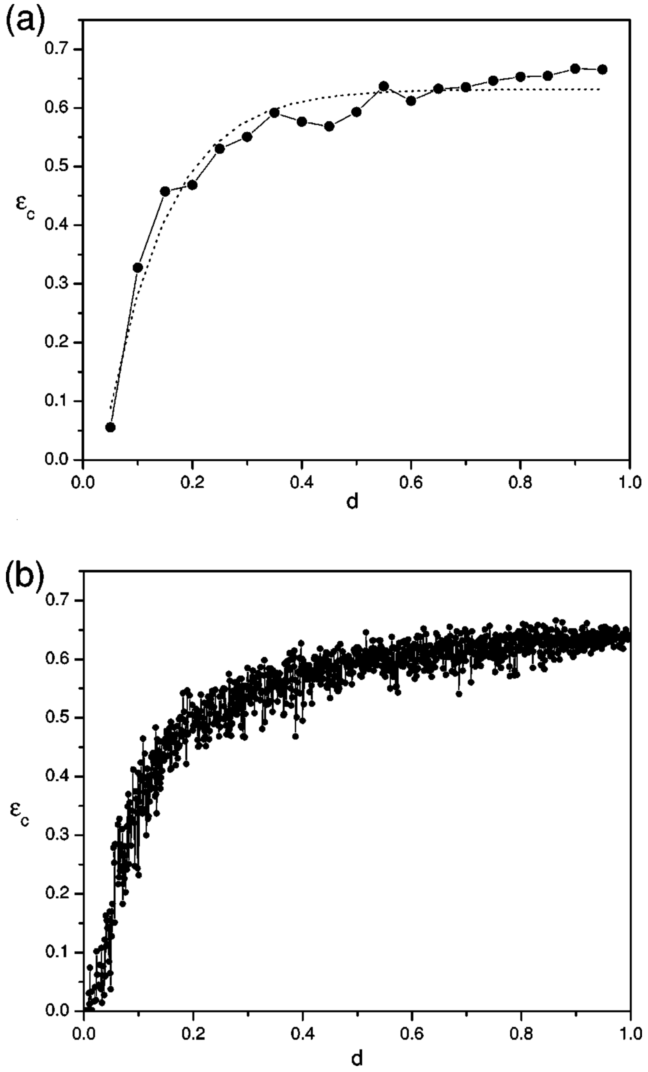

FIG. 2. The simulations of relationship between the critical coupling parameter ε_c and the linked degree d with $N=200$, corresponding to step size of linked degree (a) $\triangle d=0.05$ and (b) $\triangle d=0.001$.

It is obvious that the larger the coupling intensity ε becomes, the more easily synchronization is reached for the system, with other parameters remaining the same. Concomitantly, the question is whether there exists a critical coupling parameter ε_c, and furthermore, whether there exists a dependent relationship between ε_c and the topological structure in the subsystems. In fact, since the evolution of networks is sensitive to the varied initial states and the different random connecting weight matrices in systems with the same linked degree d, it is impossible to find an identical value of ε_c. However, the fact that the dispersion for these ε_c, corresponding to varied initial states and connecting matrices in our simulations is distributed in a narrow region and in a universal way, even as N goes into infinity, causes us to investigate the qualitative curve of ε_c vs d.

Figure 2(a) shows a plot of the critical coupling intensity ε_c versus the linked degree d in subsystems with the size $N=200$. One can see that the qualitative relation between ε_c and d is close to a sigmoidal curve. The larger d becomes, the larger ε_c becomes. This can be explained from the fact that the evolution of subsystems with larger d is more stable and it needs a more powerful coupling parameter to drive their evolutions into synchronization. In Fig. 2(b), we present the plot of the simulation with the step size of linked degree $\triangle d=0.001$ in the same conditions as Fig. 2(a). Comparing both plots of simulations, the agreement is excellent for the global tendency of the qualitative behavior of ε_c versus d.

From Fig. 3, it follows that it is more difficult to come into synchronization with an increase of the size of subsystems. It is clear, however, that there exists a homologous asymptotic behavior in the region of larger linked degree. Now, the case is how the limit of the coupling intensity depends on the varied size of networks. In Fig. 4, we present the plot of ε_c versus $1/N$ for the linked degree $d=1$, which is identical to the limit case. The form of the limit coupling intensity as a function of the inverse of the size of networks calls for a fitting of these data with an exponential decay function

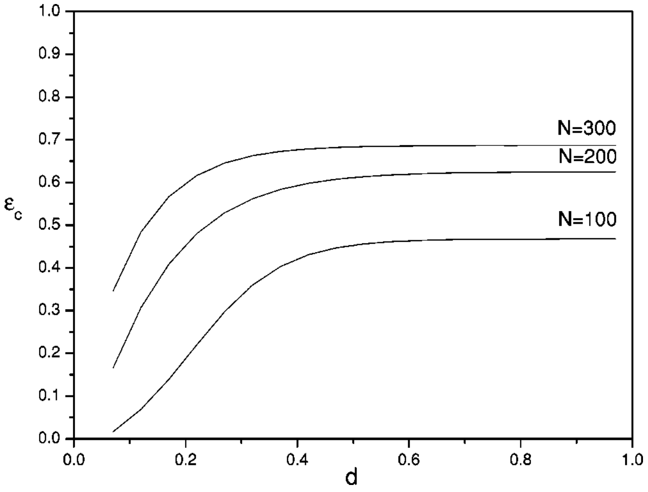

FIG. 3. The qualitative relationship of ε_c vs d for various sizes of networks N.

$$\varepsilon_c = A + Be^{-1/(CN)}, \tag{4}$$

where the constant $A=0.44\pm0.024$, $B=0.44\pm0.020$, and $C=0.0066\pm0.00071$. It follows that the maximal critical coupling intensity corresponding to $N\rightarrow\infty$ is set as 0.88 ± 0.044, and in the same sense for another extreme case, the minimal ε_c is 0.44 ± 0.024 if both global connecting subsystems designed by Eq. (1) come into synchronization.

Another important topic is the fraction of coupling neurons between two subsystems. The considered coupling system can be viewed as a structure made of two horizontal layers of networks. Apparently, from the definition, Eq. (1), of the above investigated systems, the neurons are involved in global vertical coupling interactions between two layers, or the dimensionality of coupling parameter is identical to the size of subsystems. Considering the real physical systems or the potential applications, the coupling interactions must be diluted and modified with time. As a result, the systems defined by Eq. (1) can be redefined as

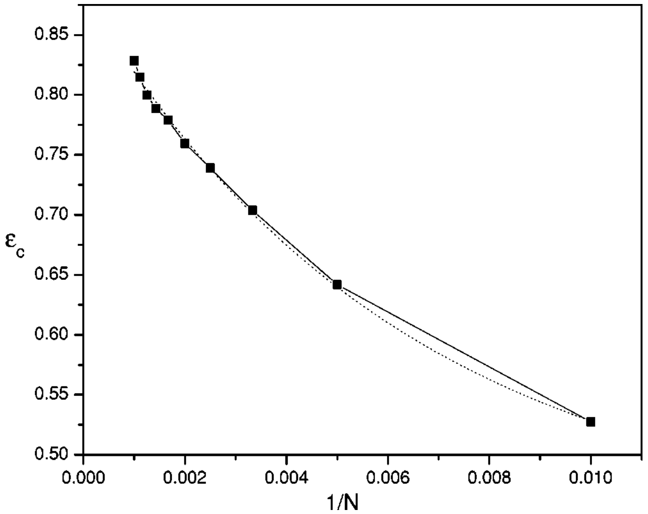

FIG. 4. The plot of ε_c vs $1/N$ for various sizes of networks in the limit case $d=1$. The exponential decay fitted curve of this relationship is shown by a dotted line.

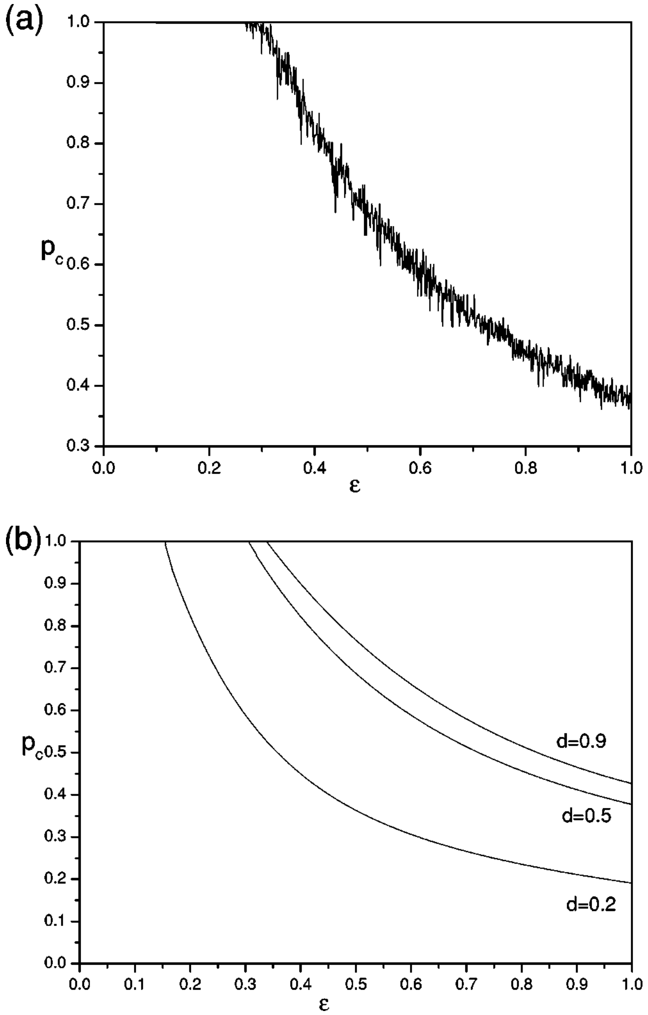

FIG. 5. The synchronization diagram of minimal coupling probability p_c vs the corresponding coupling intensity ε (a) for systems with size $N=100$, $d=0.5$ and (b) for systems with varied linked degree d and $N=100$ in the form of qualitative curves.

$$x_i^1(t+1) = [1-\varepsilon\xi_i(t)]\Theta(h_i^1(t)) + \varepsilon\xi_i(t)\Theta(h_i^1(t)+h_i^2(t)),$$
$$x_i^2(t+1) = [1-\varepsilon\xi_i(t)]\Theta(h_i^2(t)) + \varepsilon\xi_i(t)\Theta(h_i^1(t)+h_i^2(t)), \tag{5}$$

where $\xi_i(t)\in\{0,1\}$ is a random number with probability $1-p$ and p, respectively.

For revealing the association between the critical coupling fraction p_c and the coupling intensity ε, the qualitative diagram of p_c versus ε is shown based on numerical simulations for subsystems with $N=100$, $d=0.5$ [see Fig. 5(a)]. It is easy to get an acceptable conclusion that p_c decreases with the increase of ε. Note that the series of turning points corresponding to $p_c=1$ in Fig. 5(b) are equivalent to the points in the plot of ε_c versus d (cf. Fig. 3).

In addition, other valuable information about the minimal critical fraction for synchronization of this extended system denoted by Eq. (5) can be revealed from another critical

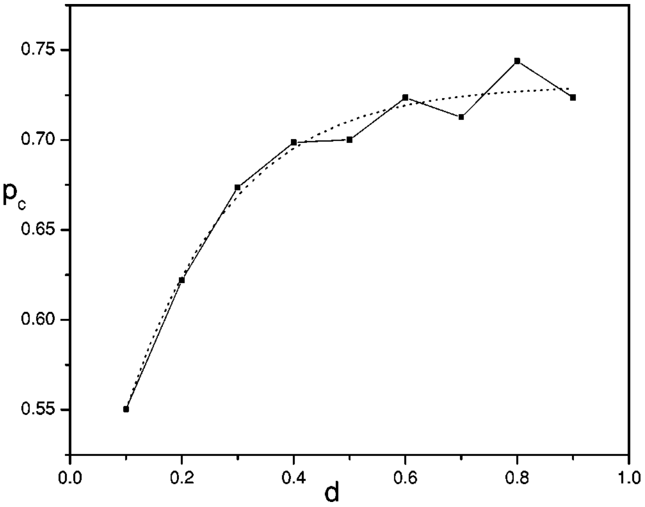

FIG. 6. The qualitative relationship of p_c vs d with the subsystem size $N=400$.

point at $\varepsilon=1$ in Fig. 5(a) and Fig. 5(b). In Fig. 6, the minimal critical coupling probability for various linked degrees of the subsystem with $N=400$ is presented. During the evolution of networks, the neuron of each site in both networks updates due to the competing effect of the local rules and the coupling mechanism. It is clear that the region above the curve is the synchronization part, while the lower part is desynchronized, thus the curve embodies a competing relationship between local correlation and stochastic coupling. It is possible to give the minimal critical coupling fraction for various linked degrees in the synchronization of this extended system with $N \to \infty$, which is more analogous to the case of real biotic systems, through analyzing curves of p_c vs d for various sizes of the subsystem. However, considering our computational capability, the more intensive and detailed work is left out for future investigation.

In this paper, we have studied the critical features of the coupling parameter in the synchronization of neural networks for various structural topologies. We obtain an exponential decay form in the case of global coupling among subsystems and full connection in each network. We find that there exists a maximal and minimal critical coupling intensity for synchronization in this extended system. For the case of partial coupling, a primary result about the critical coupling fraction for various linked degrees of networks is shown. Considering that the definition of our model is that of a typical spatially extended system, it is useful to study related topics in other extended systems, such as synchronization and directed percolation in a coupled map lattice [11], coupled ordinary differential equations, and partial differential equations.

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